A Fast Approximate Method for Calculating the Gain and Pattern of an Axisymmetric Circular Prime-fed Paraboloidal Reflector Antenna

R. Sengupta* and S.W. Ellingson.

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1 Introduction

Calculation of the gain and pattern of an axisymmetric circular prime-fed paraboloidal reflector antenna normally requires an integration over the surface of the reflector for every direction considered. However, for certain aperture field distributions, the pattern can be expressed approximately in closed form using expressions from [1] (Sec. 9.5.2). This report summarizes this method and shows how this approach can be extended to practical reflector systems which do not exhibit exactly the prescribed aperture field distributions.

This report is organized as follows: Section 2 explains the method as it appears in [1]. Section 3 explains the extended version of the method, which applies to axisymmetric circular prime-fed paraboloidal reflector antenna systems. Section 4 shows an example of the method, comparing the results to traditional physical optics (PO) surface integration.

2 Original Method

The method from [1] addresses two possible distributions of the electric field in the aperture plane: "parabolic" (referring to the magnitude of the aperture distribution) and "parabolic on a pedestal." These cases correspond to edge illumination which is either zero or non-zero at the edge of the reflector, and are addressed in Sections 2.1 and 2.2, respectively.

2.1 Zero Illumination at Reflector Edge

The normalized electric field $E_a(\rho')$ for smooth parabolic taper from the aperture center to zero at the edge is presumed to be (Figure 1):

$$E_a(\rho') = \left[1 - \left(\frac{\rho'}{a}\right)^2\right]^n \tag{1}$$

where ρ' is the radial distance from the aperture center, a is the radius of the aperture and n determines the specific distribution. The phase of the electric field is presumed to be constant in the aperture. The corresponding normalized pattern is found to be:

*Bradley Dept. of Electrical & Computer Engineering, Virginia Tech. E-mail: ramonika@vt.edu.

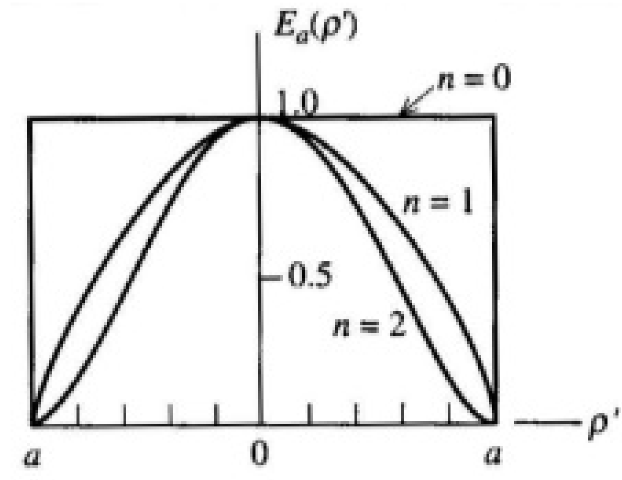

Figure 1: Parabolically-tapered electric field distribution $E_a(\rho')$ (Image Credit: [1])

n	HPBW (rad)	SLL (dB)	ε_t	$f(\theta, n)$	Distribution
0	$1.02\lambda/2a$	-17.6	1.00	$2J_1(\beta a \sin\theta)/\beta a \sin\theta$	Uniform
1	$1.27\lambda/2a$	-24.6	0.75	$8J_2(\beta a \sin\theta)/(\beta a \sin\theta)^2$	Parabolic
2	$1.47\lambda/2a$	-30.6	0.56	$48J_3(\beta a \sin\theta)/(\beta a \sin\theta)^3$	Parabolic squared

Table 1: Characteristics of the patterns obtained from the aperture field distribution shown in Figure 1 (adapted from [1]).

$$f(\theta, n) = \frac{2^{n+1}(n+1)!\ J_{n+1}(\beta a \sin\theta)}{(\beta a \sin\theta)^{n+1}} \tag{2}$$

where θ is the angle from the reflector axis of rotation, J_n is the Bessel function of the first kind and n^{th} order, $\beta = 2\pi/\lambda$ is the wavenumber, and λ is wavelength.

Table 1 shows examples for specific integer values of n, as well as the associated half-power bandwidth (HPBW), first sidelobe level (SLL), and taper efficiency ε_t. The taper efficiency can be calculated without integration using Equation 9-181 in [1] with $C = 0$, yielding:

$$\varepsilon_t = \frac{2n+1}{(n+1)^2} \tag{3}$$

For this aperture distribution, the spillover efficiency ε_s is identically 1 for all values of n, so in this case the aperture efficiency

$$\varepsilon_{ap} = \varepsilon_t \varepsilon_s \tag{4}$$

is equal to ε_t for all values of n. The on-axis gain G of the system is given by

$$G = \varepsilon_{ap} \frac{4\pi}{\lambda^2} A_{phys} \tag{5}$$

where $A_{phys} = \pi a^2$ is the physical area of the aperture. Thus:

$$G = \varepsilon_{ap} \left(\frac{\pi D}{\lambda} \right)^2 \tag{6}$$

where $D = 2a$ is the diameter of the reflector.

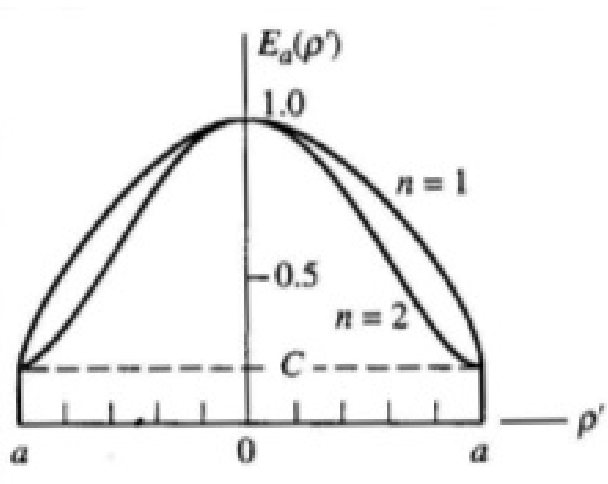

Figure 2: Parabolic-on-a-pedestal electric field amplitude $E_a(\rho')$ (Image Credit: [1]).

C (dB)	$n=1$ HPBW (rad)	SLL (dB)	ε_t	$n=2$ HPBW (rad)	SLL (dB)	ε_t
-8	$1.12\lambda/2a$	-21.5	0.942	$1.14\lambda/2a$	-24.7	0.918
-10	$1.14\lambda/2a$	-22.3	0.917	$1.17\lambda/2a$	-27.0	0.877
-12	$1.16\lambda/2a$	-22.9	0.893	$1.20\lambda/2a$	-29.5	0.834
-14	$1.17\lambda/2a$	-23.4	0.871	$1.23\lambda/2a$	-31.7	0.792
-16	$1.19\lambda/2a$	-23.8	0.850	$1.26\lambda/2a$	-33.5	0.754
-18	$1.20\lambda/2a$	-24.1	0.833	$1.29\lambda/2a$	-34.5	0.719
-20	$1.21\lambda/2a$	-24.3	0.817	$1.32\lambda/2a$	-34.7	0.690

Table 2: Characteristics of parabolic-taper-on-a-pedestal circular aperture distribution (adapted from [1]).

2.2 Non-zero Illumination at Reflector Edge

The field $E_a(\rho')$ for the parabolic-on-a-pedestal distribution (Figure 2) is presumed to be:

$$E_a(\rho') = C + (1-C)\left[1 - \left(\frac{\rho'}{a}\right)^2\right]^n \tag{7}$$

where $C = |E_a(a)|$; i.e., the magnitude of the normalized aperture field at the edge. As in [1], we refer to C as the *edge illumination.* The normalized pattern in this case is given by

$$f(\theta, n, C) = \frac{Cf(\theta, n=0) + \frac{1-C}{n+1} f(\theta, n)}{C + \frac{1-C}{n+1}} \tag{8}$$

Table 2 shows examples for specific values of n and C. The taper efficiency can be calculated without integration using Equation 9-181 in [1]:

$$\varepsilon_t = \frac{\left[C + \frac{1-C}{n+1}\right]^2}{C^2 + \frac{2C(1-C)}{n+1} + \frac{(1-C)^2}{2n+1}} \tag{9}$$

As before, the aperture efficiency $\varepsilon_{ap} = \varepsilon_t \varepsilon_s$. In this case, however, the spillover efficiency ε_s is less than 1 and [1] does not offer a method to calculate ε_s. Thus, it is not possible to calculate G using this method.

3 Extended Method

A common problem is knowing D and focal ratio f/D of the reflector, and also something (perhaps C) about the feed; from this, one desires the G and a plausible normalized pattern as a closed-form function of θ. The method described in Section 2.2 is not suitable in this case because the appropriate choice for n is not clear.

3.1 Feed Model

An appropriate choice for n can be determined if we introduce a feed model as part of the problem statement. To this end, let us assume the following normalized feed pattern:

$$f_f(\theta_f) = \begin{cases} \cos^q \theta_f & 0 \leq \theta_f \leq \pi/2 \\ 0 & \pi/2 \leq \theta_f \leq \pi \end{cases} \tag{10}$$

where θ_f is the angle measured from the reflector axis toward the vertex of the reflector, and the parameter q is used to set the gain G_f of the feed. This particular choice of feed model is chosen because (1) It is known to be representative of a broad class of practical feeds used in this application [1] (Sec. 9.6.6), and (2) Necessary numerical parameters for this feed-reflector combination are relatively easy to compute. In particular, it is known that ([1], Equation 9-230):

$$\varepsilon_s = 1 - \cos^{2q+1} \theta_0 \tag{11}$$

where θ_0 is the value of θ_f corresponding to the edge of the reflector, given by ([1], Equation 9-185b):

$$\theta_0 = 2 \arctan \left(\frac{1}{4f/D} \right) \tag{12}$$

Also, the relationship between C and q is given by ([1], Equation 9-233):

$$C = \frac{1 + \cos \theta_0}{2} \cos^q \theta_0 \tag{13}$$

which can be solved for q given C.

3.2 Determining n

What remains is to choose n to be consistent with the aperture distribution arising from the chosen feed model. This could be done in a number of different ways. Here, we propose to solve for n using Equation 9 given ε_t and C. The required input value of ε_t can obtained using Equation 4 with ε_s given by Equation 11 and ε_{ap} given (in general) by ([1], Equation 9-226):

$$\varepsilon_{ap} = \frac{G_f}{4\pi^2} \left(\cot^2 \frac{\theta_0}{2} \right) \left| \int_{\theta_f=0}^{\theta_0} \int_{\phi=0}^{2\pi} f_f(\theta_f, \phi) \tan \frac{\theta_f}{2} \, d\theta_f \, d\phi \right|^2 \tag{14}$$

where G_f is given in this case by ([1], Equation 9-231):

$$G_f = 2(2q+1) \tag{15}$$

so that Equation 14 reduces to

$$\varepsilon_{ap} = 2(2q+1)\left(\cot^2\frac{\theta_0}{2}\right)\left|\int_{\theta_f=0}^{\theta_0}\cos^q\theta_f\tan\frac{\theta_f}{2}\;d\theta_f\right|^2 \tag{16}$$

Summarizing: The proposed method to determine the appropriate value of n requires: (1) Computing ε_{ap} using Equation 16, (2) Computing ε_t using Equation 11 followed by Equation 4, and then (3) Numerically solving Equation 9 for n.

While there is no formal equality between the aperture field in the actual system and the aperture field associated with the value of n determined by this procedure, at least it is known that both aperture fields exhibit the same taper efficiency and spillover efficiency.

3.3 Revised Pattern Function

Equation 2 is not suitable for the typically non-integer values of n that will be obtained using the procedure described in the previous section. This is because it contains a factor of $(n+1)!$, and the factorial is not defined for non-integer values of n. However, the gamma function $\Gamma(\cdot)$ can be used for this purpose, since

$$(n-1)! = \Gamma(n) \tag{17}$$

Thus, Equation 2 becomes:

$$f(\theta,n) = \frac{2^{n+1}\Gamma(n+2)\;J_{n+1}(\beta a\sin\theta)}{(\beta a\sin\theta)^{n+1}} \tag{18}$$

3.4 Summary of the Method

With these considerations in mind, we now propose the following procedure for characterization of prime focus-fed circular axisymmetric reflector systems.

Given: Focal ratio f/D, feed pattern which is well-modeled as having the "$\cos^q\theta_f$" form of Equation 10, and either feed parameter q or edge illumination C.

Procedure:

1. The angle to rim θ_0 is calculated using Equation 12.
2. If not provided as part of the problem statement, q is calculated from C using Equation 13.
3. Efficiencies ε_s and ε_{ap} are calculated using Equations 11 and 16, respectively.
4. Taper efficiency $\epsilon_t = \epsilon_{ap}/\epsilon_s$. If the normalized pattern function is not desired, then stop.
5. If not provided as part of the problem statement, C is calculated from q and θ_0 using Equation 13.
6. The pattern parameter n is calculated from ε_t and C by solving Equation 9.

Upon completing this procedure, the following can now be calculated given D and frequency of operation:

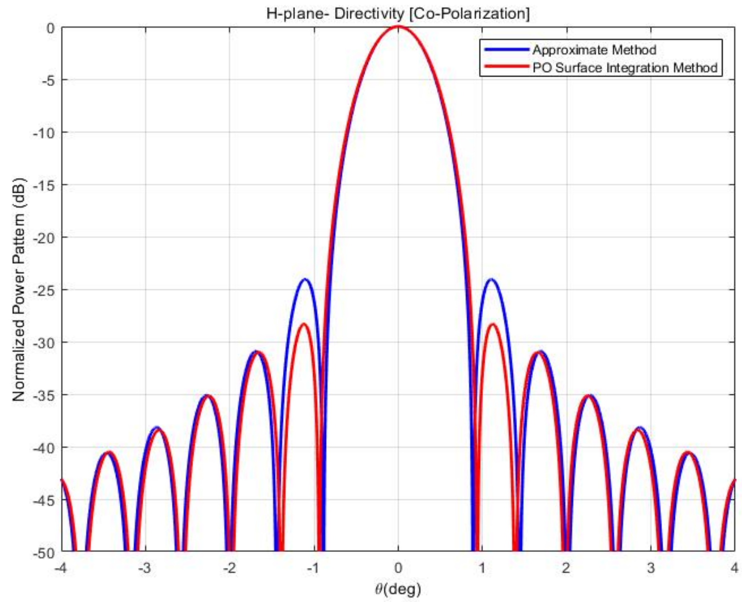

Figure 3: Normalized power pattern calculated for the example. H-plane co-polarized.

- The on-axis gain G, from Equation 5.
- The normalized power pattern $f(\theta, n, C)$ from Equation 8, using $f(\theta, n)$ from Equation 18.
- HPBW and SLL may be determined from the normalized power pattern.

4 Example

In this section we demonstrate the method outlined in Section 3.4 for a system with $f/D = 0.37$ and $C = -11$ dB. We obtain in order:

1. $\theta_0 = 68.1°$.
2. $q = 0.90$.
3. $\varepsilon_s = 0.937$, $\varepsilon_{ap} = 0.831$.
4. $\epsilon_t = 0.887$.
5. C already determined via problem statement.
6. $n = 1.28$.

The normalized power pattern is shown in Figure 3. It is found from the obtained pattern that HPBW $= 0.69°$ and SLL $= -24.1$ dB. To obtain a value for the gain, let us assume frequency of operation 11.95 GHz and $D = 2.4$ m. This yields $G = 48.5$ dBi.

As a check of the method, we now compare this result to results obtained using PO. In this case

Parameter	proposed method	PO
Gain	48.5 dBi	48.3 dBi
HPBW	$0.69°$	0.71^o
SLL	-24.1 dB	-27.0 dB

Table 3: Comparison of the proposed approximate method to the PO analysis.

we assume the same dish (i.e., same D and f/D) and same frequency. The feed is assumed to have the following form:

$$\mathbf{H}^i(s^i) = I_0 \frac{\hat{\mathbf{y}} \times \hat{\mathbf{s}}^i}{|\hat{\mathbf{y}} \times \hat{\mathbf{s}}^i|} \frac{e^{-jks^i}}{s^i} (\cos\theta_f)^q \quad , \quad \theta_f \leq \pi/2 \tag{19}$$

with q =0.90, which yields $C = -11$ dB in the principal planes. Thus, we have a plausible electromagnetic model for the feed that appears to be well-modeled in the form of Equation 10. Using PO, we obtain the pattern shown in Figure 3. It is found that $G = 48.3$ dBi, HPBW $= 0.71°$ and SLL $= -27.0$ dB. Table 3 compares the values obtained from the approximate method and the values estimated from PO. Note that the agreement is good, especially considering the incomplete knowledge employed in the proposed approximate method.

Acknowledgement

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References

[1] W.L. Stutzman and G.A. Thiele, *Antenna Theory and Design*, 3rd ed., Wiley, 2013.